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We can have sequences of objects other than real numbers, but in this course we will restrict ourselves to sequences of real numbers and will from now on just refer to sequences.

Notation

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Example: $b_n = n^2$. This sequence can also be described by $1, 4, 9, 16, 25, \ldots$.

Convergence of a Sequence

We often want to know whether the terms of a sequence $\{a_n\}$ approach some limit as $n \to \infty$. This is analogous to an ordinary limit at infinity, so we define a limit of a sequence by appropriately modifying the definition of an ordinary limit at infinity. Recall:

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$\lim_{x \to \infty} f(x) = L$ if for every $\epsilon > 0$ there is some real number N such that $|f(x) - L| < \epsilon$ whenever $x > N$.

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If a sequence has a limit, we say it converges; otherwise, we say it diverges.

Properties of Limits of Sequences

Limits of sequences share many properties with ordinary limits. Each of the following properties may be proven essentially the same way the analogous properties are proven for ordinary limits. (Each of these properties depends on the limit on the right side existing.)

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Sequences Through Ordinary Functions

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Theorem

Consider a sequence $\{a_n\}$ *and an ordinary function* f. *If* $a_n = f(n)$ *and* $\lim_{x\to\infty} f(x) = L$, *then* $\lim a_n = L$.

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Theorem

Consider a sequence $\{a_n\}$ and an ordinary function f. If $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$, then $\lim a_n = L$.

Proof.

Suppose the hypotheses are satisfied and let $\epsilon > 0$. Since $\lim_{x\to\infty} f(x) = L$, if follows there must be some $N \in \mathbb{R}$ such that $|f(x) - L| < \epsilon$ whenever $x > N$. Since $a_n = f(n)$, it follows that $|a_n - L| < \epsilon$ whenever $n > N$. $\square$

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- $\lim \dfrac{n}{2^n} = 0$

A similarly flavored limit which needs to be proven separately is $\lim \dfrac{2^n}{n!} = 0$.

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Using L'Hôpital's Rule

L'Hôpital's Rule cannot be used directly to find limits of sequences, but it can be used indirectly.

We can often find $\lim a_n$ by finding a function $f(x)$ such that $a_n = f(n)$ and then using L'Hôpital's Rule to find $\lim_{x\to\infty} f(x)$.

Example

We want to find $\lim \frac{n \ln n}{n^2+1}$. We let $f(x) = \frac{x \ln x}{x^2+1}$. We can then use L'Hôpital's Rule to find

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x \ln x}{x^2+1} = \lim_{x\to\infty} \frac{x \cdot \frac{1}{x} + (\ln x) \cdot 1}{2x}$$
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so $\lim \dfrac{n \ln n}{n^2+1} = 0.$

Monotonic Sequences

Sometimes it is possible and even necessary to determine whether a sequence converges without having to find what it converges to. This is often the case with monotonic sequences.

Definition (Increasing)

A sequence $\{a_n\}$ is increasing if $a_k \leq a_{k+1}$ for all k in its domain.

Definition (Strictly Increasing)

A sequence $\{a_n\}$ is strictly increasing if $a_k < a_{k+1}$ for all k in its domain.

Definition (Decreasing)

A sequence $\{a_n\}$ is decreasing if $a_k \geq a_{k+1}$ for all k in its domain.

Definition (Strictly Decreasing)

A sequence $\{a_n\}$ is strictly decreasing if $a_k > a_{k+1}$ for all k in its domain.

Monotonic Sequences

Definition (Monotonicity)

If a sequence is either increasing or decreasing, it is said to be monotonic.

Definition (Boundedness)

A sequence $\{a_n\}$ is said to be bounded if there is a number $B \in \mathbb{R}$ such that $|a_n| \leq B$ for all n in the domain of the sequence. B is referred to as a bound.

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The Monotone Convergence Theorem becomes very important in determining the convergence of infinite series.

The Completeness Axiom

The proof of the Monotone Convergence Theorem depends on:

The Completeness Axiom: *If a nonempty set has a lower bound, it has a greatest lower bound; if a nonempty set has an upper bound, it has a least upper bound.*

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Exercise: Write down precise definitions.

We will give a proof of the Monotone Convergence Theorem for an increasing sequence. A similar proof can be created for a decreasing sequence.

Proof of the Monotone Convergence Theorem

Proof.

If a sequence is increasing and has a limit, it is clearly bounded below by its first term and bounded above by its limit and thus must be bounded, so we'll just show that a sequence which is increasing and bounded must have a limit.

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It follows that if $n > N$, $B - \epsilon < a_N < a_n \leq B$, so $|a_n - B| < \epsilon$ and it follows from the definition of a limit that $\lim a_n = B$. □

Definition (Infinite Series)

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The terms of a series form a sequence, but in a series we attempt to *add* them together rather than simply list them.

We don't actually have to start with $k = 1$; we could start with any integer value although we will almost always start with either $k = 1$ or $k = 0$.

Convergence of Infinite Series

We want to assign some meaning to a *sum* for an infinite series. It's naturally to add the terms one-by-one, effectively getting a sum for part of the series. This is called a partial sum.

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$S_n = \sum_{k=1}^{n} a_k$ is called the n^{th} partial sum of the series $\sum_{k=1}^{\infty} a_k$.

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If the series doesn't converge, we say it diverges.

The Series 0.33333 . . .

With the definition of a series, we are able to give a meaning to a non-terminating decimal such as $0.33333\ldots$ by viewing it as $0.3+0.03+0.003+0.0003+\cdots = \frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\cdots = \sum_{k=1}^{\infty} \frac{3}{10^k}$.

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Using the definition of convergence and a little algebra, we can show this series converges to $\dfrac{1}{3}$ as follows.

The n^{th} partial sum
$S_n = \sum_{k=1}^{n} \frac{3}{10^k} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^{n-1}}$.

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Multiplying both sides by 10, we get
$10S_n = \sum_{k=1}^{n} \frac{3}{10^{k-1}} = 3 + \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^{n-2}}$.

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Subtracting, we get $10S_n - S_n = 3 - \frac{3}{10^{n-1}}$, so $9S_n = 3 - \frac{3}{10^{n-1}}$
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Clearly, $\lim S_n = \frac{1}{3}$, so the series $\sum_{k=1}^{\infty} \frac{3}{10^k}$ converges to $\frac{1}{3}$.

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We can obtain a compact formula for the partial sums as follows:

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Geometric Series

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$rS_n = ar + ar^2 + ar^3 + \ldots ar^{n-1} + ar^n$.

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We may summarize this information by noting the geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ converges to $\dfrac{a}{1-r}$ if $|r| < 1$ but diverges if $|r| \geq 1$.

Note on an Alternate Derivation

We could have found S_n differently by noting the factorization $1 - r^n = (1 - r)(1 + r + r^2 + \dots r^{n-1})$, which is a special case of the general factorization formula
$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots ab^{n-2} + b^{n-1})$.

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It immediately follows that $1 + r + r^2 + \dots r^{n-1} = \dfrac{1-r^n}{1-r}$.

Positive Term Series

Definition (Positive Term Series)

A series $\sum_{k=1}^{\infty} a_k$ is called a *Positive Term Series* is $a_k \geq 0$ for all k.

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Proof.

Looking at the sequence of partial sums,

$$S_{n+1} = \sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1} = S_n + a_{n+1} \geq S_n, \text{ since } a_{n+1} \geq 0.$$

Thus $\{S_n\}$ is monotonic and, by the Monotone Convergence Theorem, converges is and only if it's bounded. □

Note and Notation

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This is analogous to the notation used for convergence of improper integrals with positive integrands.

Example: $\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges

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Thus $0 \leq S_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{x^2}\,dx =$
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Since the sequence of partial sums is bounded, the series converges. $\square$

Estimating the Error

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We thus see estimating the series by the n^{th} partial sum leaves an error no larger than $\dfrac{1}{n}$, which can be made as small as desired by making n large enough.

Recall:

Theorem (Comparison Test for Improper Integrals)

Let $0 \leq f(x) \leq g(x)$ for $x \geq a$.

1. *If $\int_a^\infty g(x)\,dx < \infty$, then $\int_a^\infty f(x)\,dx < \infty$.*
2. *If $\int_a^\infty f(x)\,dx = \infty$, then $\int_a^\infty g(x)\,dx = \infty$.*

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The Comparison Test for Positive Term Series is used analogously to the way the Comparison Test for Improper Integrals is used.

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Notes About the Proof

1. The proof assumed $0 \leq a_k \leq b_k$ for $k \geq 1$. Since a change in any finite number of terms doesn't affect convergence, the conclusion must hold as long as $0 \leq a_k \leq b_k$ for k sufficently large.

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2. The second case is the contrapositive of the first, so it does not have to be proven separately.

Using the Comparison Test

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Geometric series have already been analyzed. P-Series are analogous to the improper integrals used with the P-Test for Convergence of Improper Integrals. The P-Test for Series can be proven using the Integral Test.

Theorem (Integral Test)

Let $f(x) \geq 0$, integrable for x large enough, monotonic and $\lim_{x\to\infty} f(x) = 0$ and suppose $a_k = f(k)$. It follows that

$$\sum_{k=1}^{\infty} a_k < \infty \text{ if and only if } \int_{\alpha}^{\infty} f(x)\,dx < \infty.$$

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Suppose the integral converges. For convenience, we will assume $\alpha = 1$. The proof can easily be modified if the integral is defined for some other α, but the argument is made most clearly without that complication.

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Since the improper integral converges, the integral on the right is bounded. Thus the sequence of partial sums is bounded and the series must converge.

If the integral diverges, we may use the observation $S_n \geq \int_1^{n+1} f(x)\,dx$ to show the sequence of partial sums is not bounded and the series must diverge. $\square$

Error Estimation

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If one estimates the sum of a series $\sum_{k=1}^{\infty} a_k$ by its n^{th} partial sum $s_n = \sum_{k=1}^{n} a_k$, the error will equal the sum $\sum_{k=n+1}^{\infty} a_k$ of the terms not included in the partial sum.

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If the series is a positive term series and $a_k = f(k)$ for a decreasing function $f(x)$, the analysis used in proving the Integral Test leads to the conclusion that this error is bounded by $\int_n^{\infty} f(x)\,dx$.

Example

Suppose we estimate the sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ by s_{100} and want a bound on the error.

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This may be easier said than done.

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So we need $\frac{1}{2n^2} \leq 10^{-8}$, which may be solved as follows:

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So we need $\frac{1}{2n^2} \le 10^{-8}$, which may be solved as follows:

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Suppose we want to estimate $\sum_{k=1}^{\infty} \frac{1}{k^3}$ to within 10^{-8}. We need to find n such that $\int_n^{\infty} \frac{1}{x^3}\,dx \le 10^{-8}$.

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$\sqrt{5 \cdot 10^7} \le n$
Since $\sqrt{5 \cdot 10^7} \approx 7071.07$, we need to add 7072 terms to estimate the sum to within 10^{-8}.

P-Test for Series

$$\sum_{k=1}^{\infty}\frac{1}{k^p}\begin{cases} < \infty & \text{if } p > 1 \\ = \infty & \text{if } p \leq 1. \end{cases}$$

Standard Series

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Geometric Series

$$\sum_{k=0}^{\infty} ar^k \begin{cases} \text{converges} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

Absolute Convergence

Definition (Absolute Convergence)

$\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

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Clearly, if this theorem wasn't true, the terminology of absolute convergence would be very misleading.

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The terms of the positive term series $\sum_{k=1}^{\infty} a_k +$ and $\sum_{k=1}^{\infty} a_k -$ are both bounded by the terms of the convergent series $\sum_{k=1}^{\infty} |a_k|$.

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It follows immediately that

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (a_k^+ - a_k^-) = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-$ also converges. □

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Definition (Conditional Convergence)

A convergent series which is not absolutely convergent is said to be conditionally convergent.

Testing for Absolute Convergence

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We will study one more test for convergence, the Ratio Test.

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The ratio test is usually stated as a test for absolute convergence, but can also be thought of as a test for convergence of positive term series. We state both versions below and use whichever version is more convenient.

Theorem (Ratio Test for Positive Term Series)

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Strategy For Testing Convergence

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- Next, check whether the series is one of the standard series, such as a P-Series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ or a Geometric Series $\displaystyle\sum_{k=1}^{\infty} ar^{k-1}$. If so, we can immediately determine whether it converges. Otherwise, we continue.

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- We start by testing for absolute convergence.

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- Find a reasonable series to compare it to. One way is to look at the different terms and factors in the numerator and denominator, picking out the largest (using the general criteria powers of logs $<<$ powers $<<$ exponentials $<<$ factorials), and replacing anything smaller than the largest type by, as appropriate, 0 (for terms) or 1 (for factors).

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- If the series seems to almost be geometric, the Ratio Test is likely to work.
- As a last resort, we can try the Integral Test.
- If the series is not absolutely convergent, we may be able to show it converges conditionally either by direct examination or by using the Alternating Series Test.

Strategy for Analyzing Improper Integrals

Essentially the same strategy may be used to analyze convergence of improper integrals.